

# THE ONE-DIMENSIONAL INVERSE WAVE SPECTRAL PROBLEM WITH DISCONTINUOUS WAVE SPEED

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## ABSTRACT.

The inverse problem for the Sturm- Liouville operator with complex periodic potential and positive discontinuous coefficients on the axis is studied. Main characteristics of the fundamental solutions are investigated, the spectrum of the operator is studied. We give formulation of the inverse problem, prove a uniqueness theorem and provide a constructive procedure for the solution of the inverse problem.

**Key words:** Discontinuous equation, Spectral singularities, Inverse spectral problem, Continuous spectrum.

**MSC:** 34A36, 34L05, 47A10, 47A70

## INTRODUCTION.

We consider the differential equation

$$-y''(x) + q(x)y(x) = \lambda^2 \rho(x)y(x) \quad (1)$$

in the space  $L_2(-\infty, +\infty)$  where the prime denotes the derivative with respect to the space coordinate and assume that the potential  $q(x)$  is of the form

$$q(x) = \sum_{n=1}^{\infty} q_n e^{inx}, \quad (2)$$

the condition  $\sum_{n=1}^{\infty} |q_n|^2 = q < \infty$  is satisfies,  $\lambda$  is a complex number, and

$$\rho(x) = \begin{cases} 1 & \text{for } x \geq 0, \\ \beta^2 & \text{for } x < 0, \beta \neq 1, \beta > 0. \end{cases} \quad (3)$$

This equation, in the frequency domain, describes the wave propagation in a nonhomogeneous medium, where  $q(x)$ - the restoring is force and  $1/\rho(x)$  is the wave speed. The discontinuities in  $\rho(x)$  correspond to abrupt changes in the properties of the medium in which the wave propagates.

If  $\rho(x) = 1$ , then equation (1) is called the potential equation. Especially, we would like to indicate that generalized Legendre equation, degenerate hyper-

geometrical equation, Bessel's equation and also Mathieu equation after suitable substitution coincide with potential equation (1)[1,p.374]-[2].

In regard to the problems with discontinuous coefficients, we remark that Sabatier and his co-workers [3-6] studied the scattering for the impedance-potential equation and the similar problems were intensively studied by many authors in different statements [7], [8], but for periodic complex potential they are considered for the first time.

Firstly potential (2) was considered by M.G.Gasymov[9]. Later in 1990 the results obtained in [8] were extended by Pastur L.A., Tkachenko V.A [10]. As a final remark we mention some related work of Guillemin, Uribe [11] and [12-14]. In this paper we will study the spectrum and also solve the inverse problem for singular non-self-adjoint operator. As the coefficient allows bounded analytic continuation to the upper half-plane of the complex plane  $z = x + it$ , we can conduct detailed analysis of problem (1)-(3).

The paper consists of three parts.

In part 1 we study the properties of fundamental system of solutions of equation (1). The spectrum of problem (1)-(3) is investigated in part 2. In part 3 we give a formulation of the inverse problem, prove a uniqueness theorem and provide a constructive procedure for the solution of the inverse problem

### **1. REPRESENTATION OF FUNDAMENTAL SOLUTIONS.**

Here we study the solutions of the main equation

$$-y''(x) + q(x)y(x) = \lambda^2 \rho(x)y(x)$$

that will be convenient in future.

We first consider the solutions  $f_1^+(x, \lambda)$  and  $f_2^+(x, \lambda)$ , determined by the conditions at infinity

$$\lim_{Imx \rightarrow \infty} f_1^+(x, \lambda) e^{-i\lambda x} = 1,$$

$$\lim_{Imx \rightarrow \infty} f_2^+(x, \lambda) e^{i\beta \lambda x} = 1.$$

We can prove the existence of these solutions if the condition  $\sum_{n=1}^{\infty} |q_n|^2 = q < \infty$  is fulfilled for the potential. This will be unique restriction on the potential and later on we'll consider it to be fulfilled.

**Theorem 1.** Let  $q(x)$  be of the form (2) and  $\rho(x)$  satisfy condition (3). Then equation (1) has special solutions of the form

$$f_1^+(x, \lambda) = e^{i\lambda x} \left( 1 + \sum_{n=1}^{\infty} \frac{1}{n + 2\lambda} \sum_{\alpha=n}^{\infty} V_{n\alpha} e^{i\alpha x} \right), \quad \text{for } x \geq 0, \quad (4)$$

$$f_2^+(x, \lambda) = e^{-i\lambda \beta x} \left( 1 + \sum_{n=1}^{\infty} \frac{1}{n - 2\lambda \beta} \sum_{\alpha=n}^{\infty} V_{n\alpha} e^{i\alpha x} \right), \quad \text{for } x < 0. \quad (5)$$

where the numbers  $V_{n\alpha}$  are determined from the following recurrent relations

$$\alpha(\alpha - n)V_{n\alpha} + \sum_{s=n}^{\alpha-1} q_{\alpha-s}V_{ns} = 0, \quad 1 \leq n < \alpha, \quad (6)$$

$$\alpha \sum_{n=1}^{\alpha} V_{n\alpha} + q_{\alpha} = 0, \quad (7)$$

and the series

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\alpha=n}^{\infty} \alpha |V_{n\alpha}| \quad (8)$$

converges.

The proof of the theorem is similar to the proof of [10] and therefore we don't cite it here.

**Remark1:** If  $\lambda \neq -\frac{n}{2}$ ,  $\lambda \neq \frac{n}{2\beta}$  and  $Im\lambda \geq 0$ , then  $f_1^+(x, \lambda) \in L_2(0, +\infty)$ ,  $f_2^+(x, \lambda) \in L_2(-\infty, 0)$ .

Extending  $f_1^+(x, \lambda)$  and  $f_2^+(x, \lambda)$  as solutions of equation (1) on  $x < 0$  and  $x \geq 0$  respectively and using the conjunction conditions

$$\begin{aligned} y(0+) &= y(0-), \\ y'(0+) &= y'(0-), \end{aligned} \quad (9)$$

we can prove the following lemma.

**Lemma 1:**  $f_1^+(x, \lambda)$  and  $f_2^+(x, \lambda)$  may be extended as solutions of equation (1) on  $x < 0$  and  $x \geq 0$ , respectively. Then we get

$$f_2^+(x, \lambda) = C_{11}(\lambda) f_1^+(x, \lambda) + C_{12}(\lambda) f_1^-(x, \lambda) \quad \text{for } x \geq 0,$$

$$f_1^+(x, \lambda) = C_{22}(\lambda) f_2^+(x, \lambda) + C_{21}(\lambda) f_2^-(x, \lambda), \quad \text{for } x < 0,$$

where

$$f_{1,2}^-(x, \lambda) = f_{1,2}^+(x, -\lambda),$$

$$C_{11}(\lambda) = \frac{W[f_2^+(0, \lambda), f_1^-(0, \lambda)]}{2i\lambda}, \quad (10)$$

$$C_{12}(\lambda) = \frac{W[f_1^+(0, \lambda), f_2^+(0, \lambda)]}{2i\lambda},$$

$$C_{22}(\lambda) = \frac{1}{\beta} C_{11}(-\lambda), \quad C_{21}(\lambda) = -\frac{1}{\beta} C_{12}(\lambda). \quad (11)$$

**Proof:** It is easy to see that equation (1) has fundamental solutions  $f_1^+(x, \lambda), f_1^-(x, \lambda)$  ( $f_2^+(x, \lambda), f_2^-(x, \lambda)$ ) for which

$$W[f_1^+(x, \lambda), f_1^-(x, \lambda)] = 2i\lambda,$$

$$W[f_2^+(x, \lambda), f_2^-(x, \lambda)] = 2i\lambda\beta,$$

is satisfied

Really, since  $W[f_1^+(x, \lambda), f_1^-(x, \lambda)]$  and  $W[f_2^+(x, \lambda), f_2^-(x, \lambda)]$  are independent of  $x$  and the functions  $f_1^+(x, \lambda), f_1^-(x, \lambda)$  and  $f_2^+(x, \lambda), f_2^-(x, \lambda)$  allow holomorphic continuation on  $x$  to upper and lower half-planes, respectively, the Wronskian coincides as  $Imx \rightarrow \infty$ . We can show that

$$\lim_{Imx \rightarrow \infty} f_1^{\pm(j)}(x, \lambda) e^{\mp i\lambda x} = (\pm i\lambda)^j \quad j = 0, 1, \quad (12)$$

$$\lim_{Imx \rightarrow \infty} f_2^{\pm(j)}(x, \lambda) e^{\mp \lambda x} = (\pm i\lambda\beta)^j \quad j = 0, 1. \quad (13)$$

So that

$$W[f_1^+(x, \lambda), f_1^-(x, \lambda)] = 2i\lambda,$$

$$W[f_2^+(x, \lambda), f_2^-(x, \lambda)] = 2i\lambda\beta.$$

Then each solution of equation (1) may be represented as linear combinations of these solutions.

$$f_2^+(x, \lambda) = C_{11}(\lambda) f_1^+(x, \lambda) + C_{12}(\lambda) f_1^-(x, \lambda) \quad \text{for } x \geq 0.$$

$$f_1^+(x, \lambda) = C_{22}(\lambda) f_2^+(x, \lambda) + C_{21}(\lambda) f_2^-(x, \lambda), \quad \text{for } x < 0,$$

Using the conjunction conditions (9) it is easy to obtain the relation (10-11).

Let

$$f_n^{\pm}(x) = \lim_{\lambda \rightarrow \mp \frac{n}{2}} (n \pm 2\lambda) f_1^{\pm}(x, \lambda) = \sum_{\alpha=n}^{\infty} V_{n\alpha} e^{i\alpha x} e^{-i\frac{n}{2}x}, \quad (14)$$

It follows from relation (6) that if  $V_{nn} \neq 0$ , then  $V_{n\alpha} \neq 0$  for all  $\alpha > n$  and therefore  $f_n^{\pm}(x) \neq 0$ . Consequently, the points  $\pm \frac{n}{2}$ ,  $n \in N$  are not singular points for  $f_1^{\pm}(x, \lambda)$ .

Then  $W[f_n^{\pm}(x), f_1^{\mp}(x, \mp \frac{n}{2})] = 0$  and consequently the functions  $f_n^{\pm}(x), f_1^{\mp}(x, \mp \frac{n}{2})$ , that are solutions of equation (1) for  $\lambda = \pm \frac{n}{2}$ , are linear dependent.

Therefore

$$f_n^{\pm}(x) = V_{nn} f_1^{\mp}\left(x, \mp \frac{n}{2}\right), \quad (15)$$

## 2.1. SPECTRUM OF OPERATOR $L$ .

Let  $L$  be an operator generated by the operation  $\frac{1}{\rho(x)} \left\{ -\frac{d^2}{dx^2} + q(x) \right\}$

in the space  $L_2(-\infty, +\infty, \rho(x))$ .

To study the spectrums of the operator  $L$  at first we calculate the kernel of the resolvent of the operator  $(L - \lambda^2 I)$  by means of general methods.

To construct the kernel of the resolvent of operator  $L$ , we consider the equation

$$-y''(x) + q(x)y(x) = \lambda^2 \rho(x)y(x) + f(x).$$

Here,  $f(x)$  is an arbitrary function belonging to  $L_2(-\infty, +\infty)$ . Divide the plane  $\lambda$  into sectors

$$S_k = \{k\pi < \arg \lambda < (k+1)\pi\}, k = 0, 1.$$

When  $\lambda \in S_0$ , we note that every solution of equation (1) can be written in the form

$$y(x, \lambda) = C_1(x, \lambda) f_1^+(x, \lambda) + C_2(x, \lambda) f_2^+(x, \lambda). \quad (16)$$

Using the method of variation of constant, we obtain that

$$C_1'(x, \lambda) = -\frac{1}{W[f_1^+, f_2^+]} \rho(x) f_2^+(x, \lambda) f(x)$$

$$C_2'(x, \lambda) = \frac{1}{W[f_1^+, f_2^+]} \rho(x) f_1^+(x, \lambda) f(x)$$

By virtue of the condition  $y(x, \lambda) \in L_2(-\infty, +\infty)$ , we find that

$$C_2(\infty, \lambda) = C_1(-\infty, \lambda) = 0.$$

Consequently, we have

$$C_1(x, \lambda) = \int_{-\infty}^x \frac{1}{W[f_1^+, f_2^+]} \rho(t) f_2^+(t, \lambda) f(t) dt$$

$$C_2(x, \lambda) = - \int_x^{\infty} \frac{1}{W[f_1^+, f_2^+]} \rho(t) f_1^+(t, \lambda) f(t) dt.$$

Substitute them in (16) we get

$$y(x, \lambda) = \int_{-\infty}^{\infty} R_{11}(x, t, \lambda) \rho(t) f(t) dt$$

where

$$R_{11}(x, t, \lambda) = \frac{1}{W[f_1^+, f_2^+]} \begin{cases} f_1^+(x, \lambda) f_2^+(t, \lambda) & \text{for } t < x \\ f_1^+(t, \lambda) f_2^+(x, \lambda) & \text{for } t > x \end{cases} \quad \lambda \in S_0. \quad (17)$$

Calculating analogously we can construct the kernel of the resolvent on the sector  $S_1$ , namely

$$R_{12}(x, t, \lambda) = \frac{1}{W[f_1^-, f_2^-]} \begin{cases} f_1^-(x, \lambda) f_2^-(t, \lambda) & \text{for } t < x \\ f_1^-(t, \lambda) f_2^-(x, \lambda) & \text{for } t > x \end{cases} \quad \lambda \in S_1. \quad (18)$$

**Lemma 2.** The spectrum of the operator  $L$  consist of continuous spectrum filling in the axis  $\{0 \leq \lambda < +\infty\}$  on which there may exist spectral singularities coinciding with the numbers  $\frac{n}{2\beta}, \frac{n}{2}, n = 1, 2, 3, \dots$

**Proof:**

It follows from (17-18) that the resolvent exists for all complex values of  $\lambda$  and may have poles on the real axis. Let's investigate these poles. First, we'll prove that the operator  $L$  has no negative eigenvalues. Assume opposite. Let  $k < 0$  be eigenvalue of the operator  $L$  with corresponding eigenfunction. Then from (17-18) it follows that the number  $\lambda_0 = |k_0|^{1/2} \exp(i\pi/2) = i|k_0|^{1/2}$  should coincide with one of the numbers  $\frac{n}{2\beta}, \frac{n}{2}, n = \pm 1, \pm 2, \pm 3, \dots$ . The obtained contradiction proves the absence of negative eigenvalues.

Now we'll investigate the function  $R(x, t, \lambda)$  in the neighborhood of  $\lambda_0$  from  $[0, \infty)$ . Then the number  $\lambda_0$  coincides with one of the numbers  $\frac{n}{2\beta}, \frac{n}{2}, n = \pm 1, \pm 2, \pm 3, \dots$ . From (17-18) it follows that the limit  $\lim_{\lambda \rightarrow \lambda_0} (\lambda - \lambda_0) R(x, t, \lambda) = R_0(x, t)$  exists and  $R_0(x, t)$  is a bounded function with respect to all the variables. Let  $\theta(x)$  be an arbitrary finite function. Then  $\varphi(x) = \int_{-\infty}^{+\infty} R_0(x, t) \theta(t) dt$  is a bounded solution of equation (1) for  $\lambda = \lambda_0$ . Therefore  $\varphi(x) = C_0 f_1^+(x, \lambda_0)$ . Comparison of the last relation with formulae (17-18) shows that if  $\lambda_0 \neq \frac{n}{2}, \lambda_0 \neq \frac{n}{2\beta}, n \in N$  then  $C_0 = 0$  and so the kernel of the resolvent has removable singularity at the point  $\lambda_0$ . So, it remains the  $\lambda_0$  where has poles of the first order. Since  $f_1^+(x, \lambda_0) \notin L_2(-\infty, +\infty)$  then  $\lambda_0^2$  is a spectral singularity of the operator  $L$ .

Theorem is proved.

**Corollary:** The kernel  $R(x, t, \lambda)$  has no singularities on the axis  $l_1 = \{\lambda : \arg \lambda = \pi\}$ , but on the axis  $l_0 = \{\lambda : \lambda > 0\}$  it may have first order poles only at the points  $\lambda_0 = \frac{n}{2}, \lambda_0 = \frac{n}{2\beta}, n \in N$ .

**Lemma 3:** The coefficient  $C_{12}(\lambda)$  is an analytic function in the  $Im \lambda \geq 0$  and has a finite number of simple zeros, moreover, if  $C_{12}(\lambda_n) = 0$ , then

$$\frac{d}{d\lambda} C_{12}(\lambda)|_{\lambda=\lambda_n} = -i \int_{-\infty}^{+\infty} \rho(x) f_1^+(x, \lambda_n) f_2^+(x, \lambda_n) dx.$$

**Proof:**

From regularity  $W[f_1^+(x, \lambda), f_2^+(x, \lambda)]$  on  $Im \lambda \geq 0$  and use the estimation

$$W[f_1^+(x, \lambda), f_2^+(x, \lambda)] = i\lambda(\beta - 1) + O(|\lambda|^{-1})$$

we get that  $C_{12}(\lambda)$  has finite number of zeros.

Let's prove the second part of the theorem. By the standard method we can obtain from equation (1)

$$f_1^{+''}(x, \lambda) \frac{d}{d\lambda} f_2^+(x, \lambda) - f_1^+(x, \lambda) \frac{d}{d\lambda} f_2^{+''}(x, \lambda) = 2\lambda \rho(x) f_1^+(x, \lambda) f_2^+(x, \lambda)$$

Integrating this equality from  $-A$  to  $x$  we find

$$W[f_1^+(x, \lambda), \frac{d}{d\lambda} f_2^+(x, \lambda)] \Big|_{-A}^x = -2\lambda \int_{-A}^x \rho(x) f_1^+(x, \lambda) f_2^+(x, \lambda) dx. \quad (19)$$

By the analogous way we get

$$W[f_1^+(x, \lambda), \frac{d}{d\lambda} f_2^+(x, \lambda)]_x^A = -2\lambda \int_x^A \rho(x) f_1^+(x, \lambda) f_2^+(x, \lambda) dx. \quad (20)$$

On the other hand, from Lemma1 follows that

$$\frac{d}{d\lambda}(2i\lambda C_{12}(\lambda)) = W[\frac{d}{d\lambda} f_1^+(x, \lambda), f_2^+(x, \lambda)] + W[f_1(x, \lambda), \frac{d}{d\lambda} f_2^+(x, \lambda)] \quad (21)$$

Let's  $\lambda = \lambda_n$  is one of the zeros of the  $C_{12}(\lambda)$ . Comparing the formulas (19-21) we get

$$\begin{aligned} (2i\lambda \frac{d}{d\lambda} C_{12}(\lambda) + 2iC_{12}(\lambda)) \Big|_{\lambda=\lambda_n} &= 2\lambda_n \int_{-A}^A \rho(x) f_1^+(x, \lambda_n) f_2^+(x, \lambda_n) dx + \\ &+ W[\frac{d}{d\lambda} f_1^+(x, \lambda_n), f_2^+(x, \lambda_n)] \Big|_{x=-A} + W[f_1^+(x, \lambda_n), \frac{d}{d\lambda} f_2^+(x, \lambda_n)] \Big|_{x=A} \end{aligned} \quad (22)$$

As the functions  $f_1^+(x, \lambda_n)$  and  $f_2^+(x, \lambda_n)$  belong to  $L_2(-\infty, +\infty)$  at  $\lambda = \lambda_n$  therefore the second and the third addends at the right hand side in (22) are equal to zero at  $A \rightarrow +\infty$ , we find that

$$\frac{d}{d\lambda} C_{12}(\lambda) \Big|_{\lambda=\lambda_n} = -i \int_{-\infty}^{+\infty} \rho(x) f_1(x, \lambda_n) \varphi_2(x, \lambda_n) dx.$$

Lemma3 is proved

For solutions  $f_1^\pm(x, \lambda)$  and  $f_2^\pm(x, \lambda)$  we can obtain the asymptotic equalities

$$f_1^{\pm(j)}(0, \lambda) = \pm(i\lambda)^j + o(1) \text{ for } |\lambda| \rightarrow \infty, \quad j = 0, 1,$$

$$f_2^{\pm(j)}(0, \lambda) = \pm(i\lambda\beta)^j + o(1) \text{ for } |\lambda| \rightarrow \infty, \quad j = 0, 1$$

For simplicity we prove the first equality.

Since

$$f_1^\pm(0, \lambda) = 1 + \sum_{n=1}^{\infty} \sum_{\alpha=n}^{\infty} \frac{V_{n\alpha}}{n \pm 2\lambda}$$

that

$$|f_1^\pm(0, \lambda)| \leq 1 + \sum_{n=1}^{\infty} \sum_{\alpha=n}^{\infty} \frac{|V_{n\alpha}|}{|n + 2\lambda|} \leq 1 + \sum_{n=1}^{\infty} \sum_{\alpha=n}^{\infty} \frac{|V_{n\alpha}|}{\sqrt{(n + 2\operatorname{Re}\lambda)^2 + 4\operatorname{Im}^2\lambda}} \leq 1 + \frac{1}{|\operatorname{Im}\lambda|} \sum_{n=1}^{\infty} \sum_{\alpha=n}^{\infty} \frac{\alpha |V_{n\alpha}|}{n}.$$

Therefore, as  $|\lambda| \rightarrow \infty$ , we obtain  $f_1^\pm(0, \lambda) = 1 + o(1)$ . Analogously we can prove the rest asymptotic equalities as  $|\lambda| \rightarrow \infty$ . Then for the coefficients  $C_{12}(\lambda)$ ,  $C_{12}(-\lambda)$ , we get the following asymptotic equalities

$$C_{12}(\lambda) = \frac{1}{2i\lambda} (i\lambda\beta - i\lambda) + o(1) = -\frac{\beta+1}{2} + o(1), \quad (23)$$

$$C_{12}(-\lambda) = -\frac{\beta+1}{2} + o(1),$$

These asymptotic equalities and analytical properties of the coefficients  $C_{12}(\lambda)$ ,  $C_{12}(-\lambda)$  make valid the following statement.

**Lemma 4.** The eigenvalues of operator  $L$  are finite and coincide with zeros of the functions  $C_{12}(\lambda)$ ,  $C_{12}(-\lambda)$  from sectors

$$S_k = \{k\pi < \arg \lambda < (k+1)\pi\}, k = 0, 1$$

respectively.

**Remark:** Take into account (23) we can obtain the useful on later relation

$$\beta = -2 \lim_{Im\lambda \rightarrow \infty} C_{12}(\lambda) - 1 \quad (24)$$

## 2.2. EIGENFUNCTION EXPANSIONS.

**Definition 2.** The points at which resolvent have poles are called the singular numbers of operator  $L$ .

Let  $\lambda_1, \lambda_2, \dots, \lambda_l, \lambda_{l+1}, \dots, \lambda_n, \dots$  be the singular numbers of operator  $L$ . At that

$$Re\lambda_j Im\lambda_j \neq 0, \quad j = 1, 2, \dots, l$$

$$Re\lambda_j Im\lambda_j = 0, \quad j = l+1, \dots, n, \dots$$

The numbers  $\lambda_j$ ,  $j = l+1, \dots, n, \dots$  are called the spectral singularities of operator  $L$ . From the form of resolvent it is easy to see that it has singular numbers (i.e. eigenvalues)  $\lambda_1, \lambda_2, \dots, \lambda_l$  in zeros of the functions  $C_{12}(\lambda)$ ,  $C_{12}(-\lambda)$  in the sectors  $S_k = \{k\pi < \arg \lambda < (k+1)\pi\}$ ,  $k = 0, 1$  respectively. It directly follows from Lemma 2 and representation (17-18) that kernel of resolvent may have spectral singularities coinciding with the numbers  $\frac{n}{2\beta}$ ,  $\frac{n}{2}$ ,  $n = 1, 2, 3, \dots$ . Consequently taking (15) into account we calculate

$$\begin{aligned} \lim_{\lambda \rightarrow n/2} (n-2\lambda) R_{11}(x, t, \lambda) &= \lim_{\lambda \rightarrow n/2} (n-2\lambda) \frac{1}{2i\lambda} [f_1^+(x, \lambda) f_1^+(t, \lambda) \frac{W[f_2^+, f_1^-]}{W[f_1^+, f_2^+]} + \\ &+ f_1^+(x, \lambda) f_1^-(t, \lambda)] = \frac{1}{in} [V_{nn} f_1^+(x, \frac{n}{2}) f_1^+(t, \frac{n}{2}) + \\ &+ V_{nn} f_1^+(x, \frac{n}{2}) f_1^+(t, \frac{n}{2})] = \frac{2}{in} V_{nn} f_1^+(x, \frac{n}{2}) f_1^+(t, \frac{n}{2}). \end{aligned} \quad (25)$$



Analogously taking into account the denotation  $\tilde{f}_2^+(x, \lambda) = f_2^+(x, \lambda) (n - 2\lambda\beta)$ , therewith, the function  $\tilde{f}_2^+(x, \lambda)$  has no poles at the points  $\lambda = \frac{n}{2\beta}$ ,  $n \in N$ , we get

$$\begin{aligned} \lim_{\lambda \rightarrow n/2\beta} (n - 2\lambda\beta) R_{11}(x, t, \lambda) &= \lim_{\lambda \rightarrow n/2\beta} (n - 2\lambda\beta) \frac{1}{W[f_1^+, f_2^+]} \times \\ &\times [C_{22} f_2^+(x, \lambda) + C_{21}(\lambda) f_2^-(x, \lambda)] f_2^+(t, \lambda) = \lim_{\lambda \rightarrow n/2\beta} (n - 2\lambda\beta) \frac{1}{W[f_1^+, f_2^+]} \times \\ &\times \left[ -\frac{1}{\beta} \frac{W[f_2^-, f_1^+]}{2i\lambda} f_2^+(x, \lambda) + \frac{1}{\beta} \frac{W[f_1^+, f_2^+]}{2i\lambda} f_2^-(x, \lambda) \right] f_2^+(t, \lambda) = \\ &= \lim_{\lambda \rightarrow n/2\beta} (n - 2\lambda\beta) \frac{n - 2\lambda\beta}{W[f_1^+, \tilde{f}_2^+]} \left[ -\frac{1}{\beta} \frac{W[f_2^-, f_1^+]}{2i\lambda} \frac{\tilde{f}_2^+(x, \lambda)}{n - 2\lambda\beta} + \right. \\ &+ \left. \frac{1}{\beta} \frac{W[f_1^+, \tilde{f}_2^+]}{2i\lambda} \frac{1}{n - 2\lambda\beta} f_2^-(x, \lambda) \right] \frac{\tilde{f}_2^+(t, \lambda)}{n - 2\lambda\beta} = \left[ -\frac{1}{in} \frac{W[f_2^-(0, n/2\beta), f_1^+(0, n/2\beta)]}{W[f_1^+(0, n/2\beta), \tilde{f}_2^+(0, n/2\beta)]} \tilde{f}_2^+(x, n/2\beta) + \right. \\ &+ \left. \frac{W[f_1^+(0, n/2\beta), \tilde{f}_2^+(0, n/2\beta)]}{in} f_2^-(x, n/2\beta) \right] \tilde{f}_2^+(t, n/2\beta) = F(x, t) \end{aligned}$$

**Lemma 6:** Let  $\psi(x)$  be an arbitrary twice continuously differentiable function belonging to  $L_2(-\infty, +\infty, \rho(x))$ . Then

$$\int_{-\infty}^{+\infty} R(x, t, \lambda) \rho(t) \psi(t) dt = -\frac{\psi(x)}{\lambda^2} + \frac{1}{\lambda^2} \int_{-\infty}^{+\infty} R(x, t, \lambda) g(t) dt,$$

where

$$g(t) = -\psi''(x) + q(x) \psi(x) \in L_2(-\infty, +\infty).$$

Integrating the both hand side along the circle  $|\lambda| = R$  and passing to limit as  $R \rightarrow \infty$  we get

$$\psi(x) = -\lim_{R \rightarrow \infty} \frac{1}{2\pi i} \oint_{|\lambda|=R} 2\lambda d\lambda \int_{-\infty}^{+\infty} R(x, t, \lambda) \rho(t) \psi(t) dt$$

The function  $\int_{-\infty}^{+\infty} R(x, t, \lambda) \rho(t) \psi(t) dt$  is analytical inside the contour, with respect to  $\lambda$  excepting the points  $\lambda = \lambda_n, n = 1, 2, \dots, l, \lambda = \frac{n}{2}, \lambda = \frac{n}{2\beta}, n = 1, 2, \dots$ . Denote by  $\Gamma_0^+ (\Gamma_0^-)$  the contour formed by segments  $[0, \frac{1}{2\beta} - \delta], [\frac{1}{2\beta} + \delta, \frac{1}{2} - \delta], \dots, [\frac{n}{2\beta} + \delta, \frac{n}{2} - \delta]$  and semi-circles of radius  $\delta$  with the centers at points  $\frac{n}{2}, \frac{n}{2\beta}, n = 1, 2, \dots$  located in upper (lower) half plane. Then

$$\begin{aligned} \psi(x) &= -\frac{1}{2i\pi} \int_{-\infty}^{+\infty} 2\lambda \rho(t) \psi(t) [\int_{\Gamma_0^+} R_{11}(x, t, \lambda) d\lambda - \int_{\Gamma_0^-} R_{12}(x, t, \lambda) d\lambda] dt = -\frac{1}{2i\pi} \int_{-\infty}^{+\infty} 2\lambda \rho(t) \psi(t) \times \\ &\times \int_{\Gamma_0^-} [R_{11}(x, t, \lambda) - R_{12}(x, t, \lambda)] d\lambda dt + \underset{\lambda=\lambda_n}{Res} R_{11}(x, t, \lambda) + \underset{\lambda=\frac{n}{2\beta}}{Res} R_{11}(x, t, \lambda) + \underset{\lambda=\frac{n}{2}}{Res} R_{11}(x, t, \lambda) \end{aligned}$$

Separately calculate every item.

$$R_{11}(x, t, \lambda) - R_{12}(x, t, \lambda) = \frac{f_1^+(x, \lambda) f_1^+(t, \lambda)}{2i\lambda C_{12}(\lambda) C_{22}(\lambda)}$$

Residues of resolvent  $R_{11}(x, t, \lambda)$  in  $\lambda_1, \lambda_2, \dots, \lambda_l$  denote by  $G_{11}(\lambda_n)$ . Then  $G_{11}(\lambda_n)$  will be equal to

$$G_{11}(\lambda_n) = \lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) R_{11}(x, t, \lambda).$$

Then for every function  $\psi(x)$  belonging to  $L_2(-\infty, +\infty, \rho(x))$  we get following eigenfunction expansion in the form

$$\begin{aligned} \psi(x) = & -\frac{1}{2i\pi} \int_{-\infty}^{+\infty} \rho(t) \psi(t) \left[ \oint_{\Gamma_0} \frac{f_1^+(x, \lambda) f_1^+(t, \lambda)}{2i\lambda C_{12}(\lambda) C_{22}(\lambda)} d\lambda + \right. \\ & \left. + G_{11}(\lambda_n) + \frac{2}{in} V_{nn} f_1^+(x, \frac{n}{2}) f_1^+(t, \frac{n}{2}) + F(x, t) \right] dt \end{aligned} \quad (26)$$

### **SOLUTION OF THE INVERSE PROBLEM.**

Let's study the inverse problem for the problem (1-3). In spectral expansion (26) the numbers  $V_{nn}$  play a part of normalizing numbers for the function  $f_1^+(x, \lambda)$  responding it spectral singularities.

The inverse problem is formulae as follows.

**INVERSE PROBLEM.** Given the spectral data  $\{C_{12}(\lambda), V_{nn}\}$  construct the  $\beta$  and potential  $q(x)$ .

Using the results obtained above we arrive at the following procedure for solution of the inverse problem.

1. Taking into account (6) we get

$$V_{n, \alpha+n} = V_{nn} \sum_{m=1}^{\alpha} \frac{V_{m\alpha}}{m+n},$$

from which all numbers  $V_{n\alpha}$ ,  $\alpha = 1, 2, \dots$ ,  $n = 1, 2, \dots, n < \alpha$  are defined.

3. Then from recurrent formula (6)-(8), find all numbers  $q_n$ .
4. The number  $\beta$  is defined from equality

$$\beta = -2 \lim_{Im\lambda \rightarrow \infty} C_{12}(\lambda) - 1.$$

So inverse problem has a unique solution and the numbers  $\beta$  and  $q_n$  are defined constructively by the spectral data.

**Theorem 2.** The specification of the spectral data uniquely determines  $\beta$  and potential  $q(x)$ .

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